Quintic Curves for which $P = 1$.[*]

BY PETER FIELD.

The general equation of a quintic curve with five fixed double points contains five arbitrary constants. If, therefore, the curve is required to pass through five additional points, it will be completely determined.

Let ϕ_0 and ϕ_1 be two conics through the points 1, 2, 3, 4, U_0 and U_1 nodal cubics through the same four points and having a node at a point 5. Then the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$, where λ is an arbitrary constant, represents a quintic curve having nodes at the five given points. Moreover this equation is general. For, suppose $S = 0$ to be the equation of any quintic curve having the five given double points. The curve $S = 0$ intersects the conic $\phi_0 = 0$ in a pair of points (aside from the nodal points). Take U_1 through these points. Similarly pass U_0 through the two points of intersection (i. e. the two points which are not nodal points on S) of S and ϕ_1 and determine λ so that the coordinates of an additional point on $S = 0$ satisfy the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$. The only restriction in the selection of this last point is that it must not be the twenty-fifth point of intersection of two quintic curves having 1, 2, 3, 4, 5 as nodal points and the four points of $S\phi_0$ and $S\phi_1$, which are ordinary points on S, as ordinary points. The equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$ must then be identical with the equation $S = 0$.

Just as in the case of the unicursal curves, if λ is small, the curve is very nearly of the form of $\phi_0 U_0$ with the exception of two breaks at the two intersections which do not lie on the curve. Four combinations of these two breaks are possible, and it is easily seen that all of them are permissible; for (Fig. 1) the way in which the break is made at 6 can be changed by simply changing the sign of

[*] For a partial list of the forms of the non-singular quintic curves, see Bancroft, American Journal of Mathematics, Vol. X.

λ. Suppose now that the generating point moves from 6 toward 7 following very closely to ϕ_0. The way in which the break at 7 is made will depend on which side of ϕ_0 the generating point approaches 7, but this is arbitrary as the quintic crosses ϕ_0, aside from the nodal points, only at the two remaining points of intersection of U_1 and ϕ_0, and these are arbitrary. The equation might equally well have been put in a variety of other forms.

As in the case of the unicursals, a few additional forms are given by considering also the case of a quartic and a straight line, and for the same reason, viz. that while the equation $\phi_0 U_0 - \lambda\phi_1 U_1 = 0$ is general, we obtain only the forms for small values of λ and do not obtain the transitional forms. This case,

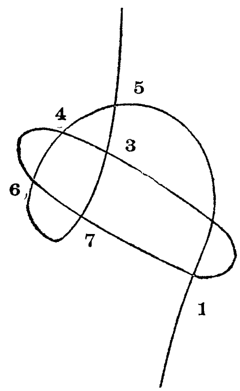

Fig. 1.

however, differs from the case of the unicursals in that there is no difficulty in writing the equation in the form $\alpha_0 U_0 - \lambda\alpha_1 U_1 = 0$ where the α's are linear and the U's quartic functions in the variables.

The method that has been used for determining the forms of quintic curves having five crunodes may by a slight modification, be used for determining the forms of curves having one or more cusps. For one cusp $\phi_0 U_0$ can be taken with two consecutive intersections and one of the breaks made at one of these points. For two cusps $\phi_0 U_0$ can be taken with an additional pair of consecutive intersections and the second break made at this point; for three cusps the preceding only needs to be modified by taking U_0 a cuspidal cubic. In case four cusps are desired, U_0 can be taken a cuspidal cubic and $\phi_0 U_0$ with three consecutive intersections at one point and two at another, the breaks now being made at the second of the three consecutive points and at one of the two consecutive points.

If U_0 is a cuspidal cubic and U_0 and ϕ_0 have three consecutive intersections at two different points and the breaks are made in each case at the second of the three points, a quintic curve with five cusps is obtained. Two pairs of these cusps have their vertices approaching coincidence, while the above curve, with four cusps, has one such pair. In each case the breaks are, of course, supposed to be made in such a way as to give cusps.*

The equation of a quintic curve having four cusps, can be written down at once as follows: Let α, α_1 be the lines and ϕ, ϕ_1 the conics represented in the accompanying figure, then the equation $\alpha\phi^2 - \lambda\alpha_1\phi_1^2 = 0$ represents a quintic

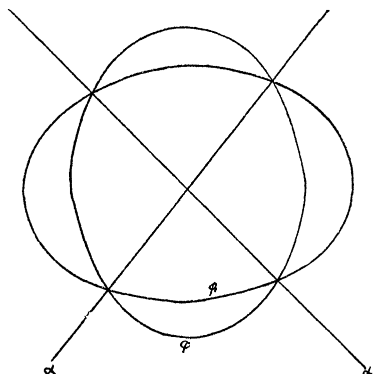

Fig. 2.

curve having cusps at the four points $\phi\phi_1$. The equation also contains a sufficient number of constants to fix another double point. Each of the conics passes through the four cuspidal points and, in addition, is tangent to the quintic at two of these points. The above equation, therefore, does not represent the general quintic curve with four cusps, as no such conics can be drawn in the general case.

The acnodal forms might also have been examined as in every case where a cusp is admissible; an acnode is also possible.†

Since the curves are not all unipartite, the sequence of the double points can not be used as a basis of classification ‡ in the same way as was done for the

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* The justification for making a break at a specified one of the consecutive points, rests on making the break while the points are distinct, and then considering them as neighboring points.

† See R. Gentry, On the Forms of Plane Quartic Curves, pp. 27, dissertation, Bryn Mawr, 1896.

‡ For an explanation of this method, see Meyer, Anwendungen der Topologie auf die Gestalten der Algebraischen Curven, Muenchen, 1878; or Tait, Edinburg Transactions, 1876-77.

unicursals,* but this scheme will be modified as follows: curves which are unipartite or which have all the nodes on one circuit, will be regarded as similar in case they have the same sequence of double points. In other cases, they will be regarded as similar in case the two branches of one curve have the same sequence of double points as those of another, as for instance 2a and 2c. In case curves, which are classed the same, differ considerably in appearance, several figures will be given.

Fig. 33 is worthy of special notice, it being the only form which permits five cusps. It has twenty real bitangents and five real inflexions. In case the loops are replaced by cusps, it has no bitangents, but it still has five real inflexions.

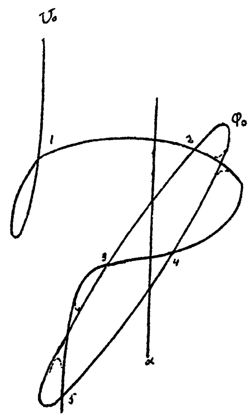


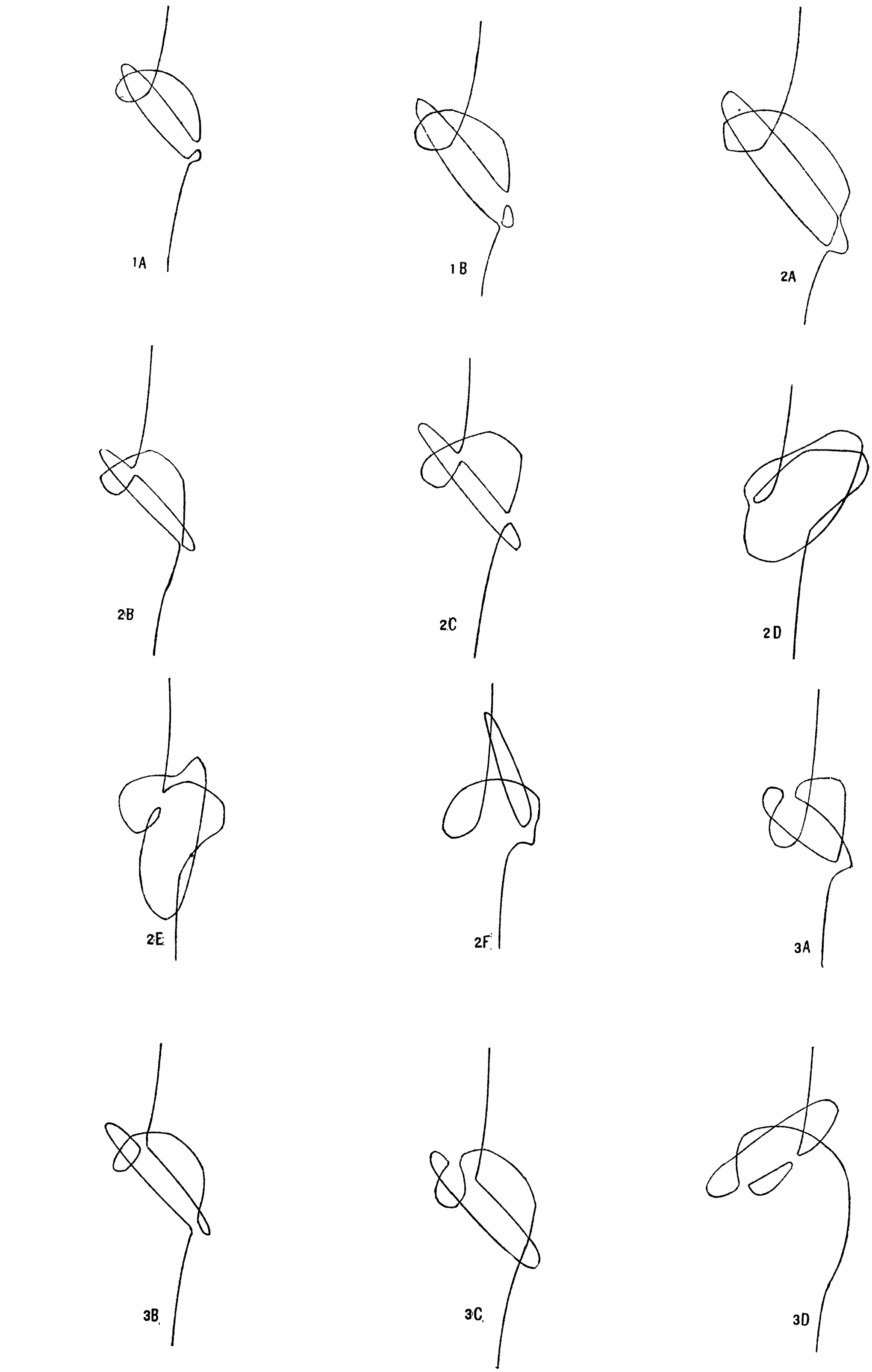
Fig. 3.

The equation of such a curve can be written down at once. Let ϕ_0, U_0, α be the curves designated in the figure; also, let ϕ_1 be a conic through the points 1, 2, 3, 4, 5 and λ a small constant, then the equation $\phi_0 U_0 - \lambda\alpha\phi_1^2 = 0$ represents a curve of the form given in Fig. 33. The sign of λ must be taken so that the breaks occur in the way indicated in the figure.

A table giving the sequence of the double points for the various forms is added. In case the curve is bipartite, the sequence of the double points for the two parts is separated by a dash.

* American Journal of Mathematics, Vol. XXVI (1904).

PLATE I.



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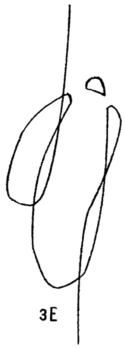

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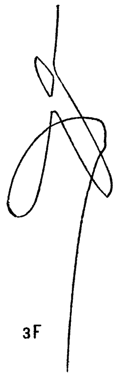

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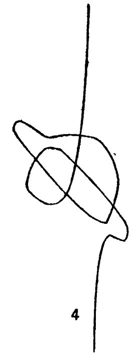

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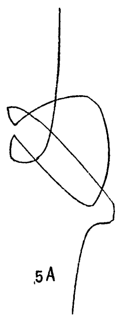

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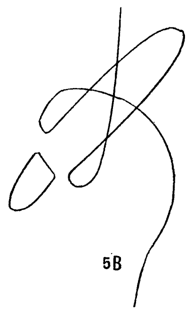

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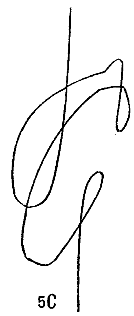

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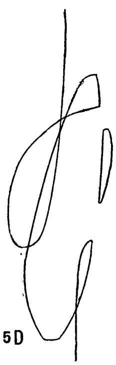

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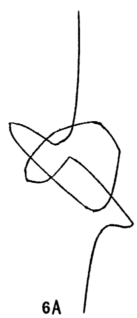

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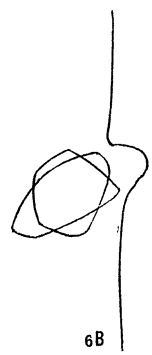

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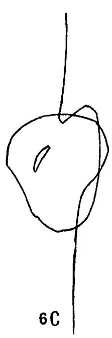

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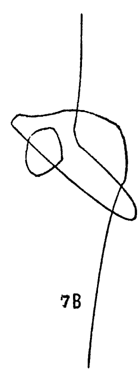

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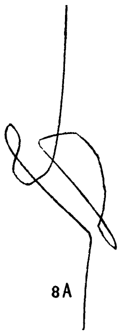

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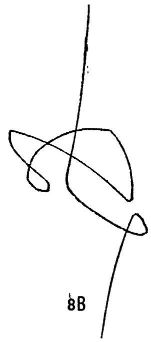

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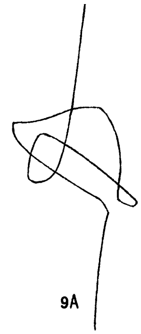

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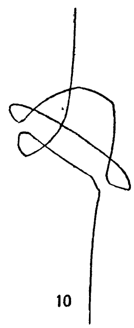

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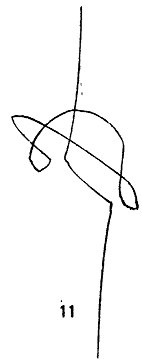

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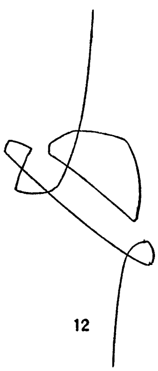

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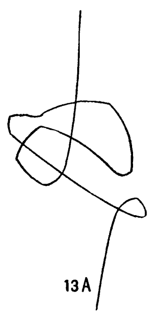

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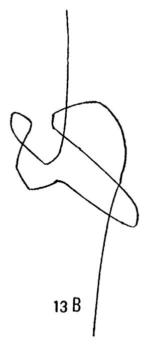

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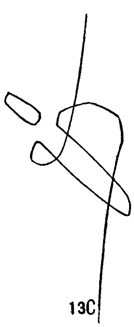

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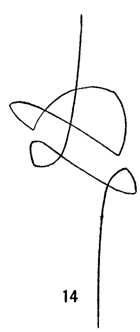

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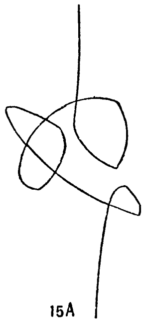

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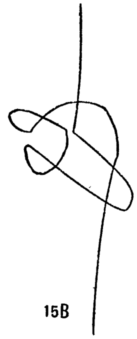

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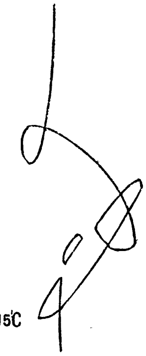

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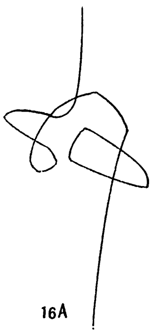

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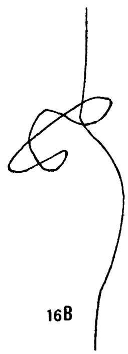

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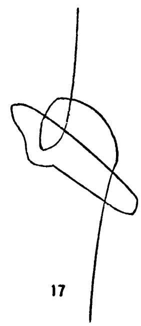

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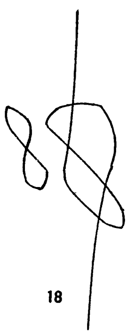

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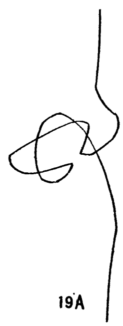

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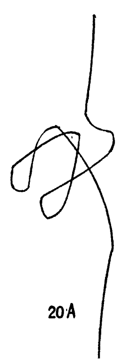

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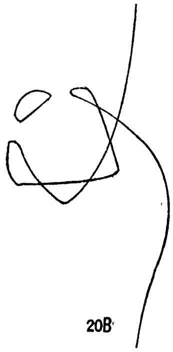

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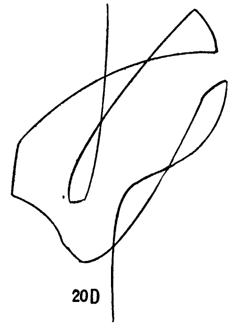

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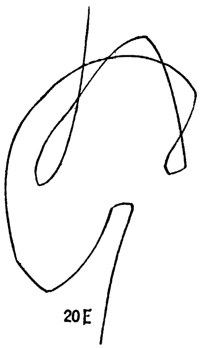

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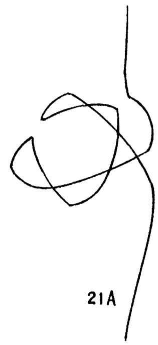

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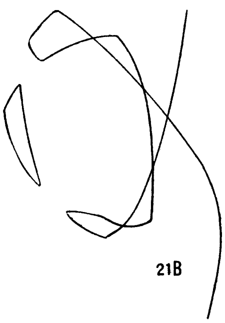

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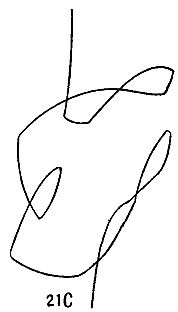

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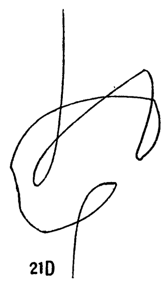

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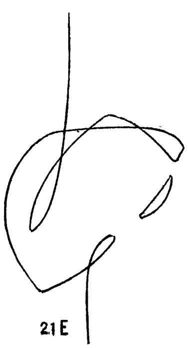

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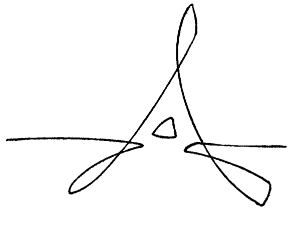

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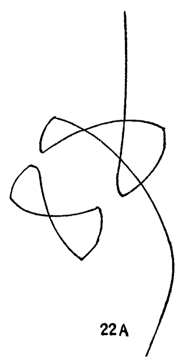

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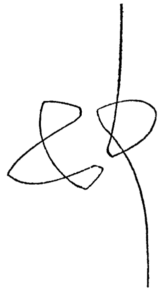
22 B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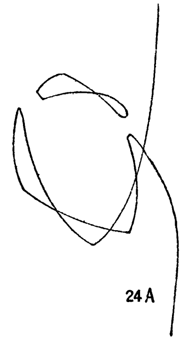
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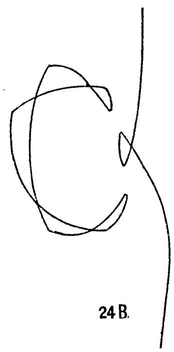
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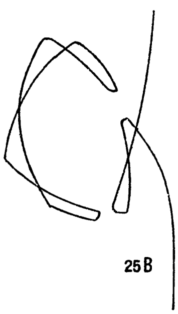
24C


25 A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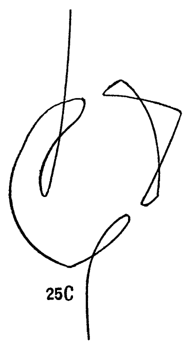
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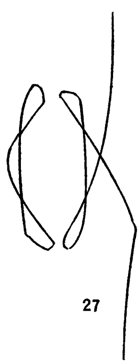
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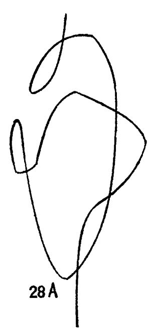
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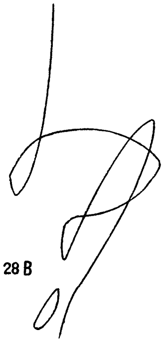

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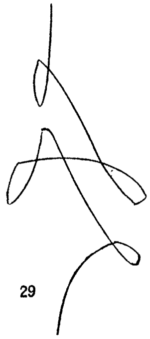

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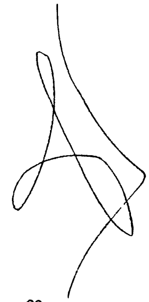

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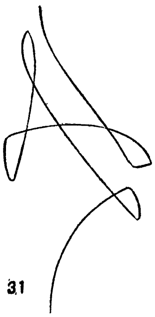

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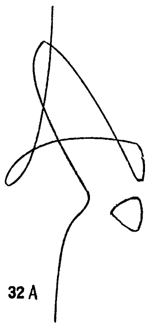

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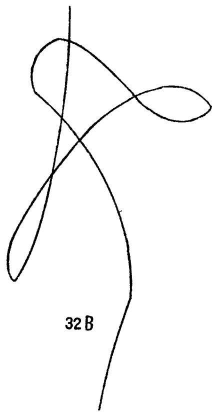

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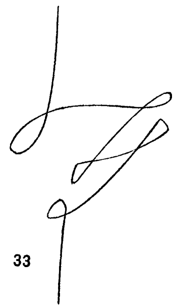

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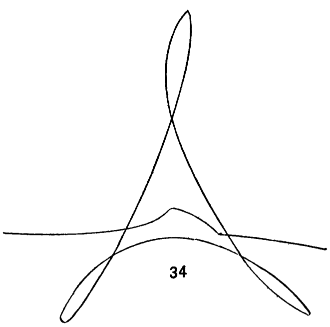

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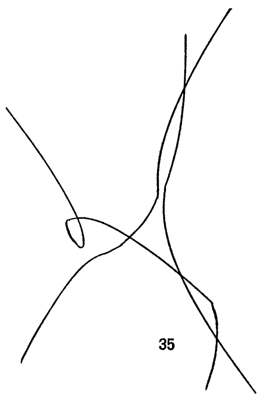

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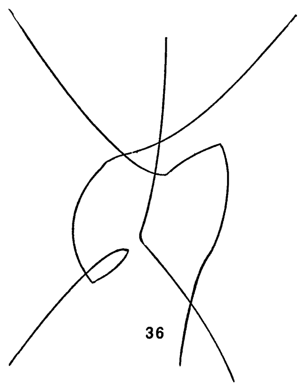

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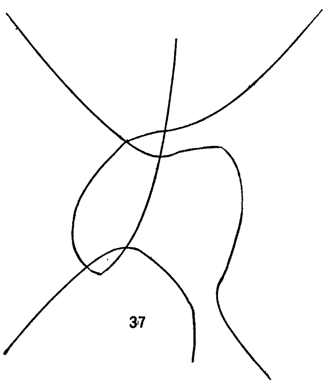

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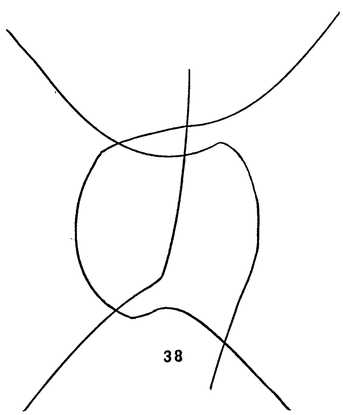

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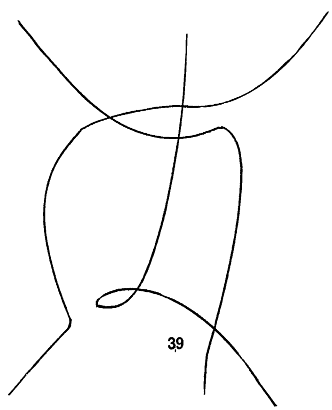
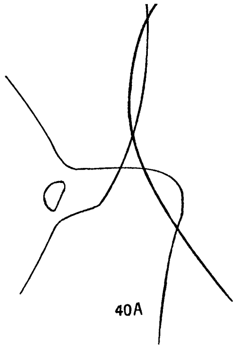
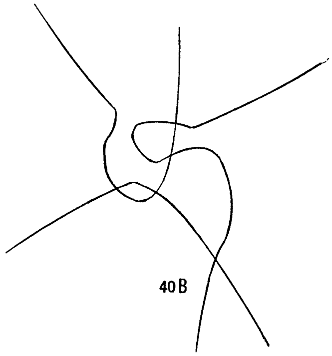


39

PLATE VIII.

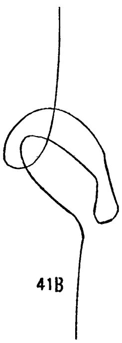


40A

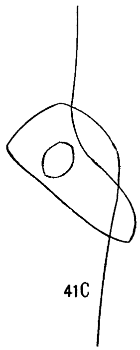


40B

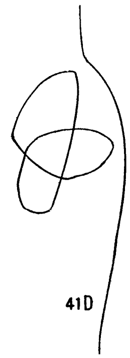
41A



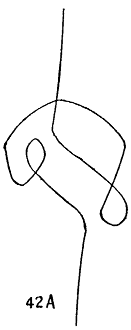
41B



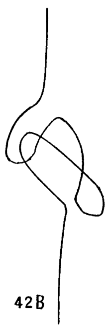
41C



41D

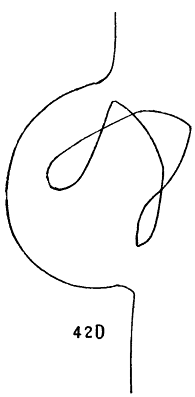


42A

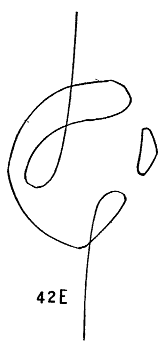


42B

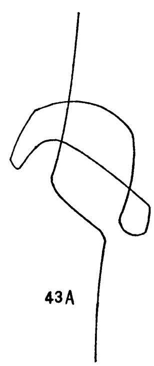
42C



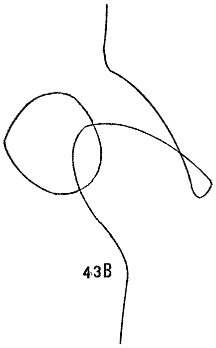
42D



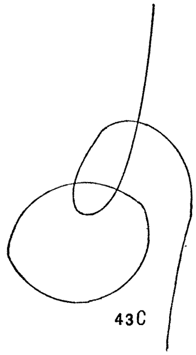
42E



43A

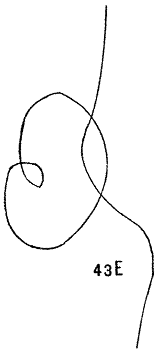


43B

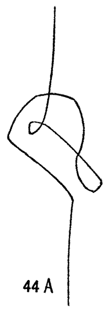


43C

43D



43E



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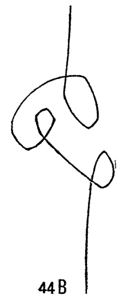
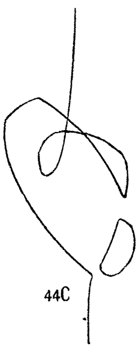
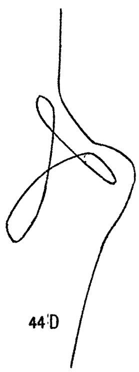


44B

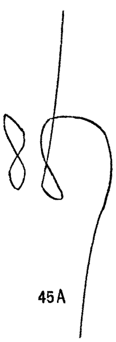


44C

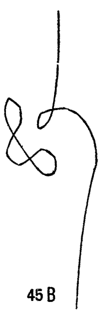


44D

44E



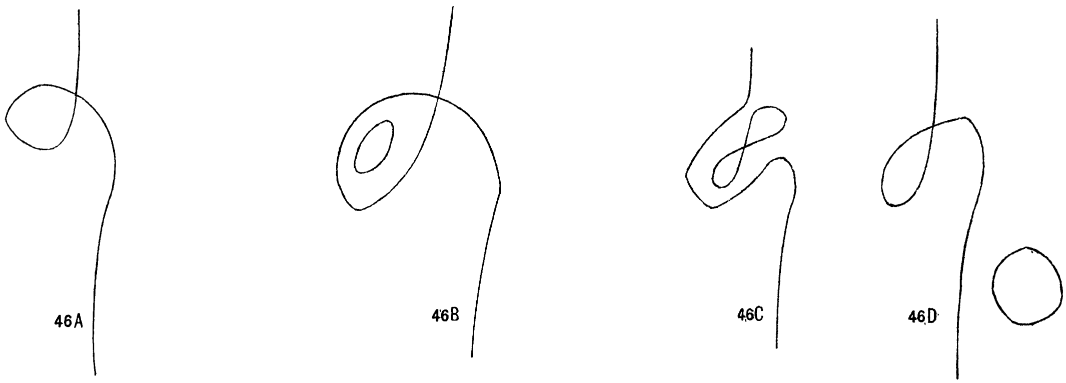
45A



45B

45C

PLATE X.



Forms 41-45 have two imaginary double points; 46 has four.

1	*abcdeabedc.*	24	*aabcddcb — ee.*
2	*abcdea — bcde.*	25	*aabccb — ddee.*
3	*aabcdecdeb.*	26	*aabccbdd — ee.*
4	*abcdec — abed.*	27	*aabbcc — ddee.*
5	*aabcddebce.*	28	*aabcdeebcd.*
6	*abcdeabcde.*	29	*aabbccdeed.*
7	*abcabcde — de.*	30	*aabbcdec — de.*
8	*aabcdb — eecd.*	31	*aabccddbee.*
9	*aabcdbedce.*	32	*aabccdebde.*
10	*aabccd — eebd.*	33	*aabbccddee.*
11	*aabcddce — eb.*	34	*aabbcdde — ce.*
12	*aabcdeeb — cd.*	35	*aabcdecd — be.*
13	*aabcdecbed.*	36	*bedced — aabc.*
14	*aabbcd — eecd.*	37	*abcdeadc — eb.*
15	*aabcdbcdee.*	38	*abcadcedbe.*
16	*abccbade — de.*	39	*aabcdbed — ce.*
17	*abcedc — abed.*	40	*abcdecdabe.*
18	*abcdbadc — ee.*	41	*abcabc.*
19	*aabcdeedcb.*	42	*aabccb.*
20	*aabcddeecb.*	43	*aabbcc.*
21	*aabccbdeed.*	44	*aabc — bc.*
22	*aabcdbcd — ee.*	45	*aabb — cc.*
23	*abcabc — ddee.*	46	*aa.*

UNIVERSITY OF MICHIGAN, *October*, 1903.